2023-24 MATH2048: Honours Linear Algebra II

Homework 6 Answer

Due: 2023-10-30 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$ with respective standard ordered bases β and γ. Define $T : V \to W$ by
$$T(p(x)) = (p(0) - 2p(1), p(0) + p'(0)),$$
where $p'(x)$ is the derivative of $p(x)$.

   (a) For $f \in W^*$ defined by $f(a, b) = a - 2b$, compute $T^*(f)$.

   (b) Compute $[T]_\beta^\gamma$ and $[T^*]_{\gamma^*}^{\beta^*}$ independently.

Solution.

(a)

$$\begin{aligned} T^*(f)(p(x)) &= f \circ T(p(x)) \\ &= f(T(p(x)) \\ &= f(p(0) - 2p(1), p(0) + p'(0)) \\ &= p(0) - 2p(1) - 2(p(0) + p'(0)) \\ &= -p(0) - 2p(1) - 2p'(0) \end{aligned}$$

(b) Let $\beta = \{1, x\}$ and $\gamma = \{e_1, e_2\}$. And $\beta^* = \{f_1, f_2\}$, $\gamma^* = \{g_1, g_2\}$ be the dual bases of β and γ.

$$[T]_\beta^\gamma = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\begin{cases} f_1(1) = 1 \\ f_1(x) = 0 \end{cases} \implies f_1(a+bx) = a, \quad \begin{cases} f_2(1) = 0 \\ f_2(x) = 1 \end{cases} \implies f_1(a+bx) = b$$

$$\begin{cases} g_1(e_1) = 1 \\ g_1(e_2) = 0 \end{cases} \implies g_1((a,b)) = a, \quad \begin{cases} g_2(e_1) = 0 \\ g_2(e_2) = 1 \end{cases} \implies g_2((a,b)) = b$$

$$\begin{aligned} T^*(g_1)(a+bx) &= g_1 \circ T(a+bx) \\ &= g_1((-a-2b, a+b)) \\ &= -a-2b \\ &= -f_1(a+bx) - 2f_2(a+bx) \text{ for any } a+bx \in P_1(\mathbb{R}) \end{aligned}$$

So $[T^*(g_1)]_{\beta^*} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$$\begin{aligned} T^*(g_2)(a+bx) &= g_2 \circ T(a+bx) \\ &= g_2((-a-2b, a+b)) \\ &= a+b \\ &= f_1(a+bx) + f_2(a+bx) \text{ for any } a+bx \in P_1(\mathbb{R}) \end{aligned}$$

So $[T^*(g_2)]_{\beta^*} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$[T^*]_{\gamma^*}^{\beta^*} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

2. Let $V = P_n(F)$, and let $c_0, c_1, \ldots, c_n$ be distinct scalars in F.

   (a) For $0 \le i \le n$, define $f_i \in V^*$ by $f_i(p(x)) = p(c_i)$. Prove that $\{f_0, f_1, \ldots, f_n\}$ is a basis for V^*.

   (b) Show that there exist unique polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ such that $p_i(c_j) = \delta_{ij}$ for $0 \le i \le n$. (Hint: Lagrange Polynomials)

   (c) For any scalars $a_0, a_1, \ldots, a_n$ (not necessarily distinct), find the polynomial $q(x)$ of degree at most n such that $q(c_i) = a_i$ for $0 \le i \le n$ and show that $q(x)$ is unique.

Solution.

(a) Consider $\sum_{i=0}^{n} a_i f_i = 0$. Let $p_i(x) = \prod_{j=0, j\neq i}^{n}(x - c_j)$, then $p_i(c_j) = 0$ iff $i = j$.

$\sum_{i=0}^{n} a_i f_i(p_j) = 0 \implies \sum_{i=0}^{n} a_i p_j(c_i) = 0 \implies a_i p_j(c_i) = 0 \implies a_j = 0$

Therefore $\{f_0, ..., f_n\}$ is linearly independent in V.

Since $|\{f_0, ..., f_n\}| = n + 1 = dim(V)$, $\{f_0, ..., f_n\}$ is a basis for V.

(b) Let $\{\hat{p}_0, ..., \hat{p}_n\} \subset V^*$ be the dual basis of $\{f_0, ..., f_n\}$.

Then $\delta_{ij} = \hat{p}_i(f_j) = f_j(p_i) = p_i(c_j)$.

The Lagrange Polynomial $p_i(x) = \frac{\prod_{j=0,j\neq i}^{n}(x-c_j)}{\prod_{j=0,j\neq i}^{n}(c_i-c_j)}$ satisfies $p_i(c_j) = \delta_{ij}$.

If $p_i(c_j) = q_i(c_j) = \delta_{ij}$ for some polynomial p_i and q_i, then $(p_i - q_i)(c_j) = 0, \forall j = 1, ..., n$. That means the polynomial $p_i - q_i$ has $n + 1$ distinct zeros. Thus $p_i - q_i = 0$. Therefore such $\{p_i\}_{i=0}^{n}$ is unique.

(c) Since $\{p_0, ..., p_n\}$ is a basis for $P_n(F)$, for any $q \in P_n(F)$, q can be uniquely represented by $q(x) = \sum_{i=0}^{n} a_i p_i(x)$ and $q(c_i) = \sum_{j=0}^{n} a_j p_j(c_i) = \sum_{j=0}^{n} a_j \delta_{ij} = a_i$.

3. Let $A, B \in M_{n\times n}(\mathbb{C})$.

(a) Prove that if B is invertible, then there exists a scalar $c \in \mathbb{C}$ such that $A + cB$ is not invertible. Hint: Examine $\det(A + cB)$.

(b) Find nonzero 2×2 matrices A and B such that both A and $A+cB$ are invertible for all $c \in \mathbb{C}$.

Solution.

(a) $\det(A+cB) = \det(B(B^{-1}A+cI)) = \det(B)\det(B^{-1}A+cI) = \det(B) f_{B^{-1}A}(-c)$.

Since $f_{B^{-1}A}(t)$, the characteristic polynomial of $B^{-1}A$, splits over $\mathbb{C}$, there exists $c \in \mathbb{C}$ such that $f_{B^{-1}A}(-c) = 0$, which implies $\det(A + cB) = 0$ and thus $A + cB$ is not invertible.

(b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ Then $\det(A + cB) = 1$ for all $c \in \mathbb{C}$

4. (a) Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients from F. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(T)(x) = g(\lambda)x$. That is, x is an eigenvector of $g(T)$ with corresponding eigenvalue $g(\lambda)$.

(b) Use (a) to prove that if $f(t)$ is the characteristic polynomial of a diagonalizable linear operator T, then $f(T) = T_0$, the zero operator. (Remark: This result does not depend on the diagonalizability of T.)

Solution.

(a) Let $g(t) = \sum_{i=0}^{n} a_i t^i$, then $g(T) = \sum_{i=0}^{n} a_i T^i$.

If $T(x) = \lambda x$, then

$$g(T)(x) = \sum_{i=0}^{n} a_i T^i(x) = \sum_{i=0}^{n} a_i(\lambda^i x) = (\sum_{i=0}^{n} a_i \lambda^i)x = g(\lambda)(x)$$

(b) T is diagonalizable, then there exists β a basis consisting of eignevectors of T.

Let $\beta = \{v_1, ..., v_n\}$ and $T(v_i) = \lambda_i v_i$. Then $f(\lambda_i) = 0$ since λ_i is the eigenvalue of T.

By (a), one has $f(T)(v_i) = f(\lambda_i)v_i = 0v_i = 0$ for $i = 1, ..., n$. Thus $f(T) = T_0$.

5. Let $A \in M_{n\times n}(F)$. Recall from §5.1 Q14 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue λ, $\dim(E_\lambda) = \dim(E'_\lambda)$.

(c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

Solution.

(a) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $f_A(t) = f_{A^t}(t) = t^2$. $\lambda = 0$ is the only eigenvalue of A and A^t. The algebraic multiplicity $\mu_A(0) = \mu_{A^t}(0) = 2$.

$E_\lambda = N(A - 0I) = \{(x, y) \in \mathbb{R}^2 | y = 0\} = span(\{(1, 0)\})$

$E'_\lambda = N(A^t - 0I) = \{(x, y) \in \mathbb{R}^2 | x = 0\} = span(\{(0, 1)\})$

So $E_\lambda \neq E'_\lambda$

(b)

$$\begin{aligned}\dim(E_\lambda) &= \dim(N(A-\lambda I)) \\ &= n - rank(A-\lambda I) \\ &= n - rank((A-\lambda I)^t) \\ &= \dim(N(A^t-\lambda I)) \\ &= \dim(E'_\lambda)\end{aligned}$$

(c) If A is diagonalizable, there exists $\lambda_1, ...\lambda_r$ such that $E_{\lambda_1} \oplus ...E_{\lambda_r} = F^n$. That is, $\dim(E_{\lambda_1}) + ... + \dim(E_{\lambda_r}) = n$. By (b), $\dim(E'_{\lambda_1}) + ... + \dim(E'_{\lambda_r}) = n$. Then $E'_{\lambda_1} \oplus ...E'_{\lambda_r} = F^n$ and thus A^t is diagonalizable.